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NOTE ON EINSTEIN'S EQUATION OF AN ORBIT.

BY F. MORLEY.

The differential equation of an orbit is given in Eddington's "Report on the Relativity Theory of Gravitation"* as

$$(1) \qquad \left[\frac{d}{d\varphi}\frac{1}{r}\right]^2 = 2m/r^3 - 1/r^2 + 2m/h^2r - (1 - c^2)/h^2,$$

r and φ being the polar coördinates of the planet. Forsyth† has discussed the equation in connection with elliptic functions. The matter is of so great interest that a second elliptic handling seems worth publication.

We have, from Eddington,

$$c^2 = 1 - m/a, \qquad h^2 = ma(1 - e^2),$$

where a is the "major semi-axis" and e the eccentricity; therefore

$$1 - c^2 = m^2(1 - e^2)/h^2.$$

The number m^2/h^2 is, in Eddington's units, about 10^{-8}. Call it α, so that

$$1 - c^2 = \alpha(1 - e^2).$$

In the calculations α^2 will be neglected when lower powers are present.

Writing $x = m/r$, the equation becomes

$$(2) \qquad \left(\frac{dx}{d\varphi}\right)^2 = 2x^3 - x^2 + 2\alpha x - \alpha(1 - e^2).$$

The roots of the cubic on the right are $\frac{1}{2}$, 0, 0 when $\alpha = 0$. Hence writing $x = \frac{1}{2} + k_1\alpha, k_2\alpha, k_3\alpha$ and determining the coefficients k, the roots are

$$x_1 = \tfrac{1}{2} + 2\alpha, \qquad x_2 = \alpha(1 - e), \qquad x_3 = \alpha(1 + e).$$

To bring equation (2) into Weierstrass's form

$$\left(\frac{dp}{dv}\right)^2 = 4p^3 - g_2p - g_3$$

we write $d\varphi = \sqrt{2}dv$, $x = p + 1/6$. The invariants g_2 and g_3 are for a quartic with coefficients a, b, c, d, e

$$ae - 4bd + 3c^2$$

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* 2d Edition (1920), p. 50. Einstein, Berlin Sitzungsberichte, 1915, p. 831.

† Proc. Royal Society, series A, vol. 97, April, 1920.

and

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

Here a, b, c, d, e are respectively $0, 1, -\frac{1}{3}, \alpha, -2\alpha^2(1-e^2)$, so that, accurately,

(3) $$g_2 = \tfrac{1}{3}(1 - 12\alpha),$$

(4) $$g_3 = \tfrac{1}{27}[1 - 18\alpha + 54\alpha^2(1-e^2)].$$

Hence the discriminant Δ, or $g_2^3 - 27g_3^2$, is

(5) $$\Delta = 4e^2\alpha^2 - 64\alpha^3.$$

The roots being real, the network or lattice which gives rise to the function p is rectangular. The essence of the problem is the determination of a period-rectangle. Let $2\omega_1$ be the real positive period, $2\lambda i\omega_1$ the imaginary period, and write as usual

$$q = e^{-\pi\lambda}.$$

The number λ is positive, and it may be called the shape of the rectangle.

The invariants g_2, g_3, are connected with the periods $2\omega_1$ and $2\omega_2$ by the equations

$$g_2 = 60 \sum{}' \frac{1}{(2m_1\omega_1 + 2m_2\omega_2)^4}, \qquad g_3 = 140 \sum{}' \frac{1}{(2m_1\omega_1 + 2m_2\omega_2)^6},$$

summed for all integer pairs m_1, m_2, the pair 0, 0 excluded. For calculation the double sums are expressed as single sums or products; and the appropriate formulæ are given in works on the elliptic functions.[*] But as the proofs of the full formulæ are necessarily complicated, I shall interpolate a proof of the approximate formulæ of a kind that is at once intelligible.

We have, n being any integer,

$$\sum_{-\infty}^{\infty} \frac{1}{(x+n)^2} = \frac{\pi^2}{\sin^2 \pi x}.$$

Differentiating,

$$\sum_{-\infty}^{\infty} \frac{1}{(x+n)^3} = \frac{\pi^3 \cos \pi x}{\sin^3 \pi x}.$$

Differentiating again,

$$\sum_{-\infty}^{\infty} \frac{1}{(x+n)^4} = \pi^4 \frac{\cos^2 \pi x + \frac{1}{3}\sin^2 \pi x}{\sin^4 \pi x} = \pi^4 \left[\frac{1}{\sin^4 \pi x} - \frac{2}{3\sin^2 \pi x}\right].$$

This enables us to sum the series for g_2, when m_2 is fixed. When $m_2 = 0$, we get the first tier

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[*] They may be found in Harkness and Morley, "Theory of Functions," p. 321, formulæ 75 and 76, and p. 324, formula 82.

$$\frac{2\cdot 60}{(2\omega_1)^4}\left[1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]=\frac{15}{2\omega_1^4}\cdot\frac{\pi^4}{90}=\frac{1}{12}\left(\frac{\pi}{\omega_1}\right)^4.$$

When $m_2 = 1$, we get the second tier

$$\sum_n \frac{1}{(2\omega_2+2m_1\omega_1)^4}=\frac{1}{(2\omega_1)^4}\sum\frac{1}{(i\lambda+m_1)^4}$$

$$=\left(\frac{\pi}{2\omega_1}\right)^4\left(\frac{1}{\sin^4\pi i\lambda}-\frac{2}{3\sin^2\pi i\lambda}\right).$$

Now

$$\sin\pi i\lambda=\left(\frac{e^{-\pi\lambda}-e^{\pi\lambda}}{2i}\right)=\left(q-\frac{1}{q}\right)\Big/2i.$$

Hence the tier $m_2 = 1$ gives

$$60\left(\frac{\pi}{2\omega_1}\right)^4\left[\frac{16q^4}{(1-q^2)^4}+\frac{8q^2}{3(1-q^2)}\right]$$

or, neglecting q^4,

$$10\left(\frac{\pi}{\omega_1}\right)^4 q^2.$$

For the tier $m_2 = -1$ we get the same result. For the tiers $m_2 \equiv \pm 2$ we have to replace q by q^2 and so on. Hence the formula

$$\text{(6)}\qquad \left(\frac{\omega_1}{\pi}\right)^4 g_2=\tfrac{1}{12}+20q^2+\cdots.$$

A similar argument, with further differentiation, gives

$$\text{(7)}\qquad \left(\frac{\omega_1}{\pi}\right)^6 g_3=\tfrac{1}{216}-\tfrac{7}{3}q^2+\cdots$$

and from these

$$\text{(8)}\qquad \left(\frac{\omega_1}{\pi}\right)^{12}\Delta=q^2+\cdots.$$

From the formulæ (6) and (8) we have approximately

$$\Delta/g_2^3 \doteq 12^3q^2$$

and on the other hand from (3) and (5) we have

$$\Delta/g_2^3 \doteq 4\cdot 27\cdot e^2\alpha^2.$$

Hence approximately

$$q^2 \doteq e^2\alpha^2/4^2,$$

$$q \doteq e\alpha/4.$$

For Mercury the eccentricity e is approximately $1/5$, so that

$$q \doteq 10^{-8}/20.$$

Calculating $\pi\lambda = \log 1/q = 9 \log 10 + \log 2$, the shape λ of the rectangle of the periods turns out to be nearly 7. As v travels round the rectangle of half periods, p takes all real values, so that the radius vector

$$r = m/x = m/(p + \tfrac{1}{6})$$

also takes all real values. For the actual orbit we take v on the upper side of the rectangle, from $v = \omega_1 + \omega_2$, $p = \alpha(1 + e)$, corresponding to perihelion, to $v = \omega_2$, $p = \alpha(1 - e)$ or aphelion.

Finally, the corresponding change in φ is $\varphi = \sqrt{2}\omega_1$, where

$$\left(\frac{\omega_1}{\pi}\right)^4 g_2 = \tfrac{1}{12}$$

for q^2, being of order α^2, is here negligible. But from (3)

$$g_2 = \tfrac{1}{3}(1 - 12\alpha).$$

Hence

$$\frac{1}{4}\left(\frac{\varphi}{\pi}\right)^4 \cdot \frac{1}{3}(1 - 12\alpha) = \frac{1}{12}.$$

$$\left(\frac{\varphi}{\pi}\right)^4 = 1/(1 - 12\alpha) = 1 + 12\alpha,$$

$$\frac{\varphi}{\pi} = 1 + 3\alpha.$$

So that in passing from perihelion to aphelion the angle passed through is not π, but

$$\pi(1 + 3m^2/h^2).$$